Journal of Global Optimization (2005) 31: 471–491
DOI 10.1007/s10898-004-1322-x

Asset Pricing under Progressive Taxes and Existence of General Equilibrium

HUI HUANG
Department of Economics, Social Science Centre, The University of Western Ontario, London, Ont., Canada N6A 5C2 (e-mail: hhuang@uwo.ca)

(Received 20 June 2003; accepted in revised form 17 May 2004)

Abstract. This paper shows that the existence of general equilibrium in a two-period economy with financial markets and progressive anonymous tax system is not at all problematic, provided securities are purely financial. We explore the concepts of weakly and strongly arbitrage-free security price for return and tax system, and prove arbitrage-free asset pricing theorems without short-sale restrictions. A general equilibrium is a set of current and future prices (contingent on uncertain events) and a set of individual plans such that all markets are cleared. The existence of such an equilibrium is proved under the following conditions: continuous, weakly convex, strictly monotone, complete preferences and strictly positive endowments.

Key words: Arbitrage-free asset pricing theorems, General equilibrium, Progressive anonymous tax system

1. Introduction

In the study of effects of taxes on arbitrage opportunities, much of the work focuses on tax-arbitrage opportunities, i.e., different types of individuals have different tax schedules so that riskless profits can made by trading across individuals (e.g. Dammon and Green, 1987; Jones and Milne, 1992). Little is known about price-arbitrage under progressive anonymous taxation. Marginal tax rates are uniform across investors within anonymous tax regime. Thus differential taxation across investors is the outcome of their different financial decisions: two investors of different types, who hold the same portfolio, bear the same tax burden. If anonymous tax is linear in returns, then it is relatively easy to prove the existence of no-arbitrage prices and general equilibrium. We only need to multiply the before-tax return matrix in the literature of general equilibrium with financial markets by a factor, one minus the tax rate. However, as the anonymous taxation becomes progressive, the existence of no-arbitrage price and general equilibrium is ambiguous. The tax bill of an agent may change in different states. Thus agents may profit risklessly by trading across states of the world.

This paper studies the existence of general equilibrium with financial markets and piecewise linear (convex) anonymous taxation in a two-period model. It extends the literature of no-arbitrage asset pricing and general equilibrium with nominal financial markets (e.g. Werner, 1985; Duffie, 1987). Trading takes place in the sequence of spot markets and futures markets for securities payable in units of account. Unlimited short-selling in securities is allowed. We focus on general progressive anonymous tax system. Restrictions on tax rules and asset returns are given to get no arbitrage conditions. Then we prove the existence of general equilibrium when the government returns all its tax revenues back to agents. To emphasize the novel effects of non-linear taxation, we abstract away from other source of income (such as labor income) and other types of taxation (such as consumption taxation, capital gains taxation).

Dammon and Green (1987) provide conditions on tax schedules and asset returns that preclude tax-arbitrage and prove the existence of equilibrium prices with linear tax regime. Though they extend their results to non-linear taxation case, they use a strong definition of tax arbitrage, which requires a tax-arbitrage portfolio to generate arbitrage at all of investors' marginal rates. Their notion of no-tax-arbitrage does not rule out prices that provide an investor with arbitrage opportunities along most of his or her tax schedule. Our paper differs significantly from Dammon and Green (1987). First, this paper focuses on price-arbitrage instead of tax-arbitrage. It comes along the literature of no-arbitrage asset pricing and existence of general equilibrium, while Dammon and Green (1987) is in a different direction of research. Second, our notion of no-arbitrage rules out prices that give an investor arbitrage opportunities along all of his or her tax schedule. Prices are always "correct" in this sense. Our paper further differs from Dammon and Green (1987) in that their analysis does not include the government as part of the economy. They implicitly assume that aggregate government tax rebates/subsidies can be unbounded by excluding governments in the economy.

Jones and Milne (1992) establish the existence of general equilibrium with linear taxation for a two-period model with one consumption good when government does not react to attempt to drain government resources. Again, what we focus on is price-arbitrage under progressive taxation while Jones and Milne (1992) emphasizes on tax-arbitrage within linear tax regime. In our model, the government returns its tax revenues to the agents. It maintains a zero balance and actively redistributes income.

The study of general equilibrium problems with asset markets developed in two directions: real assets and nominal assets (Magill and Shafer, 1991). Werner (1985), Duffie (1987), and Florenzano and Gourdel (1994) study general equilibrium with incomplete nominal asset markets. This paper extends their work to the case with progressive anonymous taxation. Market frictions (including taxation) attracted much attention recently. Chen (1995) examines incentives and economic roles of financial innovation. He also studies the effectiveness of replication-based arbitrage valuation approach in frictional economies (the friction here means holding constraints). Jouini and Kallal (1995a) derive the implications of absence of arbitrage in security markets models where traded securities are subject to short-sale constraints and where the borrowing and lending rates differ. They provide the equivalent conditions of arbitrage-free securities price system. Jouini and Kallal (1995b) study the implications of arbitrage-freeness in dynamic security markets with bid-ask spreads. Pham and Touzi (1999) address the problem of characterizing no arbitrage (strictly arbitrage-free) in the presence of friction in a discrete-time financial model, and extend the fundamental theorem of asset pricing under a non-degeneracy assumption. Basak and Croitoru (2001) use the methods of stochastic analysis to study tax-arbitrage and existence of general equilibrium for a continuous-time model with one commodity and three financial assets. Zhang et al. (2002) study arbitrage-free asset pricing in a dynamic model with proportional transaction costs.

The equivalence of general equilibrium and no-arbitrage condition has been studied since early 1980s. Harrison and Kreps (1979) first formally study arbitrage in financial markets. Kreps (1981) studies arbitrage and equilibrium in economies with infinitely many commodities and presents an abstract analysis of "arbitrage" in economies that have infinite dimensional commodity space. He proves the equivalence of no arbitrage (free lunch) and general equilibrium. Subsequently many researchers study arbitrage and come to the first and second fundamental theorems in financial economics. All the research in incomplete markets uses the equivalence of arbitrage-freeness and general equilibrium. This paper also uses the equivalence. However, to our knowledge, the concept of arbitrage-freeness with progressive anonymous tax system is initially presented here.

The structure of this paper is organized as follows. Section 2 presents a model of an exchange economy with financial markets. The progressive tax is described by continuous and convex functions. Section 3 shows weakly and strongly arbitrage-free asset pricing theorems. The equilibrium existence result is proved in Section 4 for an economy with continuous, weakly convex, strictly monotone and complete preferences.

2. Model

We consider an exchange economy with anonymous taxation over two periods 0, 1 with uncertainty about the states of nature at date 1. At date 0, it is not known which state will occur and at date 1 there is a finite set of possible states of nature $\{1, \cdots, S\}$. For notional convenience, we let period 0 denote state $s = 0$ so that the set of possible states of nature is $\{0, 1, \cdots, S\}$.

At date 0, trading takes place in L consumption goods and in J securities whose returns at date 1 depend on the states of nature. At date 1, the state of nature is revealed, securities pay their returns. The government collects taxes according a tax system and transfers taxes to each individual, and consumptions take place.

There are L goods in each state $s \in \{0, 1, \cdots, S\}$. A consumption plan $x : \{0, 1, \cdots, S\} \to \mathcal{R}^L$ specifies consumption at date 0 and in each state at date 1, that is, $x = (x_0, x_1, \cdots, x_S)$, where x_s is consumption at state s. Then the commodity space is $\mathcal{R}^{L(1+S)}$.

We study the nature of general equilibrium with security markets when all the assets are nominal. At date 0 there are spot markets for L current commodities and futures markets for J securities. Each security is described by its state-dependent return at date 1. An agent holding one share of security j receives V_s^j units of account ("money") if state $s \in \{1, \cdots, S\}$ occurs, $V_s = (V_s^1, \cdots, V_s^J) \in \mathcal{R}_+^J$. Formally, there are J functions $V^j : \{1, \cdots, S\} \to \mathcal{R}_+, j = 1, \cdots, J$.

$$V = (V^1, \cdots, V^J) = \begin{pmatrix} V_1 \\ \vdots \\ V_S \end{pmatrix}$$

where $V^j = \begin{pmatrix} V_1^j \\ \vdots \\ V_S^j \end{pmatrix}$. Thus V is a $S \times J$ matrix of returns. At date 1, in each state of the nature all L commodities will be traded on spot markets.

There are I agents ($i \in \mathcal{I} = \{1, \cdots, I\}$) defined by consumption sets $\mathcal{R}_+^{L(1+S)}$, endowments $e^i \in \mathcal{R}_+^{L(1+S)}$ and rational preference relations $\succeq_i$ over $\mathcal{R}_+^{L(1+S)}$ which are continuous, weakly convex, strictly monotone and complete.

Preference assumption: For each $i \in \mathcal{I}$, the rational preference relation $\succeq_i$ on $\mathcal{R}_+^{L(1+S)}$ satisfies the following conditions:

(1) continuous: for each $x \in \mathcal{R}_+^{L(1+S)}$, the set $\{x' \in \mathcal{R}_+^{L(1+S)} | x' \succeq_i x\}$ and $\{x' \in \mathcal{R}_+^{L(1+S)} | x \succeq_i x'\}$ are closed;
(2) weakly convex: for each $x \in \mathcal{R}_+^{L(1+S)}$, the set $\{x' \in \mathcal{R}_+^{L(1+S)} | x' \succeq_i x\}$ is convex;
(3) strictly monotone: if $x \in \mathcal{R}_+^{L(1+S)}$ and $x^+ \in \mathcal{R}_+^{L(1+S)} \setminus \{0\}$, then $x + x^+ \succ_i x$;
(4) complete: if $x_1, x_2 \in \mathcal{R}_+^{L(1+S)}$, then either $x_1 \succeq_i x_2$ or $x_2 \succeq_i x_1$.

The good price $p : \{0, 1, \cdots, S\} \to \mathcal{R}_+^L$ specifies spot price at date 0 and in each state at date 1, that is, $p = (p_0, p_1, \cdots, p_S) \in \mathcal{R}_+^{L(1+S)}$, where p_s is price at state s. Let $q \in \mathcal{R}_+^J$ denote the vector of asset prices and $\theta^i = (\theta_1^i, \cdots, \theta_J^i) \in \mathcal{R}^J$ denote the number of each of the J assets purchased by agent i (where $\theta_j^i < 0$ means short-selling asset j). The ith agent's wealth at date 0 is $p_0 e_0^i$, which is used to consume goods $(p_0 x_0^i)$ and purchase assets $(q\theta^i)$. Then his budget constraint at date 0 is

$$p_0(x_0^i - e_0^i) + q\theta^i \leqslant 0$$

The ith agent's wealth at state of nature $s \in \{1, \cdots, S\}$ is $p_s e_s^i$. The portfolio θ^i has payoff $V_s\theta^i \in \mathcal{R}$ if state $s \in \{1, \cdots, S\}$ occurs. The taxation that the agent has to pay is $f_s(\theta^i)$, which is defined later,[1] if state $s \in \{1, \cdots, S\}$ occurs. Then the government gets a total tax revenue $G_s = \sum_{i=1}^I f_s(\theta^i)$ if state $s \in \{1, \cdots, S\}$ occurs. We assume that the transfer share is $\lambda \in \mathcal{R}^I$, where $\sum_{i=1}^I \lambda_i = 1$ and $\lambda_i \geqslant 0,\ i = 1, \cdots, I$. Then the agent's transfer at state of nature $s \in \{1, \cdots, S\}$ is $\lambda_i G_s$. Therefore his total income at state of nature s is $p_s e_s^i + V_s\theta^i - f_s(\theta^i) + \lambda_i G_s$. Thus his budget constraint at date 1 is

$$p_s(x_s^i - e_s^i) \leqslant V_s\theta^i - f_s(\theta^i) + \lambda_i G_s, \quad s = 1, \cdots, S$$

Therefore the budget constraints can be written as

$$\begin{cases} p_0(x_0^i - e_0^i) + q\theta^i \leqslant 0 \\ p_s(x_s^i - e_s^i) \leqslant V_s\theta^i - f_s(\theta^i) + \lambda_i G_s, \quad s = 1, \cdots, S \end{cases}$$

and the opportunity set of agent i who buy $\theta^i = (\theta_1^i, \cdots, \theta_J^i) \in \mathcal{R}^J$ units of the J assets is given by

$$B_i(p,q) = \left\{ (x,\theta) \in \mathcal{R}_+^{L(1+S)} \times \mathcal{R}^J \,\middle|\, \begin{array}{l} p_0(x_0 - e_0^i) + q\theta \leqslant 0 \\ p_s(x_s - e_s^i) \leqslant V_s\theta - f_s(\theta) + \lambda_i G_s, \quad s = 1, \cdots, S \end{array} \right\}$$

For $s = 1, \cdots, S$, we define taxation item $f_s(\theta^i)$ as follows. There exist a set of strictly increasing constants $C_s^1, \cdots, C_s^K$ such that the marginal tax rate is t_s^k when the ith agent's return $C_s^k \leqslant V_s\theta^i < C_s^{k+1}$ for $k = 0, 1, \cdots, K$, where $C_s^0 = -\infty$ and $C_s^{K+1} = \infty$. That is to say,

$$t_s(\theta^i) = \begin{cases} t_s^0, & \text{if } V_s\theta^i < C_s^1 \\ t_s^1, & \text{if } C_s^1 \leqslant V_s\theta^i < C_s^2 \\ \vdots & \vdots \\ t_s^{K-1}, & \text{if } C_s^{K-1} \leqslant V_s\theta^i < C_s^K \\ t_s^K, & \text{if } C_s^K \leqslant V_s\theta^i \end{cases}$$

We consider the progressive tax, then $0 = t_s^0 < t_s^1 < \cdots < t_s^K$. For later use, we denote the tax system $\{t_s^0, \cdots, t_s^K\}$ as $\{t_s\}$.

Define

$$A_s^k = \{\theta \in \mathcal{R}^J | C_s^k \leqslant V_s\theta < C_s^{k+1}\} \quad \text{for } k = 0, 1, \cdots, K-1$$

where $C_s^0 = -\infty$ and $C_s^{K+1} = \infty$, that is,

$$A_s^0 = \{\theta \in \mathcal{R}^J | V_s\theta < C_s^1\}$$
$$A_s^k = \{\theta \in \mathcal{R}^J | C_s^k \leqslant V_s\theta < C_s^{k+1}\} \text{ for } k = 1, \cdots, K-1$$

[1] Werner (1985) considered the case of no frictions.

$$A_s^K = \{\theta \in \mathcal{R}^J | C_s^K \leqslant V_s\theta\}$$

Then

$$t_s(\theta) = \sum_{k=0}^{K} t_s^k 1_{A_s^k}(\theta)$$

Define tax function as follows:

$$f_s(\theta) = \begin{cases} 0, & \text{if } \theta \in A_s^0 \\ \sum_{k'=0}^{k-1}(C_s^{k'+1} - C_s^{k'})t_s^{k'} + (V_s\theta - C_s^k)t_s^k, & \text{if } \theta \in A_s^k, \quad k = 1, \cdots, K \end{cases}$$

Then f_s is a continuous convex function by definition. If $C_s^1 = +\infty$, there is no tax; if $C_s^1 = 0$ and $C_s^2 = +\infty$, there is linear tax.

An allocation bundle of agent i consists of the consumption and portfolio (x^i, θ^i), then an allocation bundle consists of the consumptions and portfolios $(x^1, \cdots, x^I; \theta^1, \cdots, \theta^I)$. Price system consists of the prices of goods and securities (p, q). We can now define the concept of a spot-financial market equilibrium as follows.

DEFINITION 1.[2] A spot-financial market equilibrium (SF equilibrium) is an allocation bundle $(\bar{x}^1, \cdots, \bar{x}^I; \bar{\theta}^1, \cdots, \bar{\theta}^I)$ and a price system $(\bar{p}, \bar{q})$ such that

(1) $(\bar{x}^i, \bar{\theta}^i) \in B_i(\bar{p}, \bar{q})$ for $i = 1, \cdots, I$;
(2) if $(x, \theta) \in B_i(\bar{p}, \bar{q})$, then $\bar{x}^i \succeq_i x$;
(3) (Spot Markets Clearance) $\sum_{i=1}^I(\bar{x}^i - e^i) = 0$;
(4) (Security Markets Clearance) $\sum_{i=1}^I \bar{\theta}^i = 0$.

3. Arbitrage-Free Asset Pricing

In this section, we study arbitrage-free asset pricing theorems with progressive anonymous tax system.

DEFINITION 2. A security price q is weakly arbitrage-free (WAF) for return V and tax system $\{t_s\}$ if any portfolio $\theta \in \mathcal{R}^J$ of securities has a positive total cost or gain $q\theta \geqslant 0$ whenever it has a positive payoff in every state $V\theta - f(\theta) \in \mathcal{R}_+^S$ (that is, $V_s\theta - f_s(\theta) \geqslant 0$ for all $s = 1, \cdots, S$).

[2]This model studies the case that tax revenue transfers to consumers. If there is no tax return $G_s = 0$, then the budget constraints can be written as

$$\begin{cases} p_0(x_0^i - e_0^i) + q\theta^i \leqslant 0 \\ p_s(x_s^i - e_s^i) \leqslant V_s\theta^i - f_s(\theta^i), \quad s = 1, \cdots, S \end{cases}$$

Thus Spot Markets Clearance (3) in Definition 1 becomes as $\sum_{i=1}^I(\bar{x}_0^i - e_0^i) = 0$ and $\sum_{i=1}^I(\bar{x}_s^i - e_s^i) \leqslant 0$ for $s \in \{0, 1, \cdots, S\}$.

DEFINITION 3.[3] A security price q is strongly arbitrage-free (SAF) for return V and tax system $\{t_s\}$ if (1) any portfolio $\theta \in \mathcal{R}^J$ of securities has a positive non-zero total cost or gain $q\theta > 0$ whenever it has a positive non-zero payoff in every state $V\theta - f(\theta) \in \mathcal{R}^S_+ \setminus \{0\}$ (that is, $V_s\theta - f_s(\theta) \geqslant 0$ for all $s = 1, \cdots, S$ and a positive non-zero payoff, at least, in one state); and (2) any portfolio $\theta \in \mathcal{R}^J$ of securities has a positive total cost or gain $q\theta \geqslant 0$ whenever it has a positive payoff in every state $V\theta - f(\theta) \in \mathcal{R}^S_+$ (that is, $V_s\theta - f_s(\theta) \geqslant 0$ for all $s = 1, \cdots, S$).

Definition 2 means if you want positive payoffs in every state at date 1, you need a positive investment at date 0. There is no free lunch. Besides the meanings of Definition 2, Definition 3 also means if you want strictly positive payoffs at date 1, you need a strictly positive investment at date 0.

In the frictionless model (Duffie, 2001), we define the marketed subspace $\{(-q\theta, V\theta) \in \mathcal{R}^{1+S} | \theta \in \mathcal{R}^J\}$ to prove arbitrage-free asset pricing theorems. In our model with progressive taxes, we cannot use the corresponding marketed "subspace" $\{(-q\theta, V_s\theta - f_s(\theta) | \theta^i \in \mathcal{R}^J\}$. Instead, we define "marketed set" as follows

$$M \equiv \left\{ y \in \mathcal{R}^{1+S} \middle| \begin{array}{ll} y_0 = -q\theta & \\ y_s \leqslant V_s\theta - f_s(\theta), & s = 1, \cdots, S \end{array} \quad \text{for } \theta \in \mathcal{R}^J \right\}$$

An element in the marketed set consists of costs you pay today ($-q\theta$, the minus sign means the decrease of your wealth) and payoffs you can get in every state tomorrow (which is less or equal to $V_s\theta - f_s(\theta)$).

LEMMA 1. *M is a closed convex set.*

Proof. Suppose $y^n \in M$ and $\lim_{n\to\infty} y^n = y$, then we want to prove that $y \in M$. Since $y^n \in M$, there exists a sequence $\{\theta^n\} \in \mathcal{R}^J$ such that $y_0^n = -q\theta^n$ and $y_s^n \leqslant V_s\theta^n - f_s(\theta^n), s = 1, \cdots, S$. $\lim_{n\to\infty} y^n = y$, so there exists a subsequence of θ^n that converges to θ. Without loss of generality, we may assume that $\{\theta^n\}$ converges to some θ. Because f_s is a continuous function, we have $y_s = \lim_{n\to\infty} y_s^n \leqslant \lim_{n\to\infty} V_s\theta^n - f_s(\theta^n) = V_s\theta - f_s(\theta)$. Therefore $y \in M$ and M is closed.

Now we prove that M is a convex set. Suppose that $y^n \in M$, $n = 1, 2$. We want to prove that, for $\gamma \in [0, 1]$, $\gamma y^1 + (1 - \gamma)y^2 \in M$. From $y^n \in M$,

[3]This definition has an equivalent form as follows. A security price q is strongly arbitrage-free (SAF) for return V and tax system $\{t_s\}$ if (1) any portfolio $\theta \in \mathcal{R}^J$ of securities has a positive non-zero total cost or gain $q\theta > 0$ whenever it has a positive non-zero payoff in every state $V\theta - f(\theta) \in \mathcal{R}^S_+ \setminus \{0\}$ (that is, $V_s\theta - f_s(\theta) \geqslant 0$ for all $s = 1, \cdots, S$ and a positive non-zero payoff, at least, in one state); and (2) any portfolio $\theta \in \mathcal{R}^J$ of securities has a zero total cost or gain $q\theta = 0$ whenever it has a zero payoff in every state $V\theta - f(\theta) = 0$ (that is, $V_s\theta - f_s(\theta) = 0$ for all $s = 1, \cdots, S$).

we have $y^n \in \mathcal{R}^{1+S}$ and there exists a $\theta^n \in \mathcal{R}^J$ such that $y_0^n = -q\theta^n$ and $y_s^n \leqslant V_s\theta^n - f_s(\theta^n), s = 1, \cdots, S, n = 1, 2$. So

$$\gamma y_0^1 + (1-\gamma)y_0^2 = -\gamma q\theta^1 - (1-\gamma)q\theta^2 = -q[\gamma\theta^1 + (1-\gamma)\theta^2]$$

and

$$\gamma y_s^1 + (1-\gamma)y_s^2 \leqslant \gamma[V_s\theta^1 - f_s(\theta^2)] + (1-\gamma)[V_s\theta^2 - f_s(\theta^2)] = V_s[\gamma\theta^1 + (1-\gamma)\theta^2] - [\gamma f_s(\theta^1) + (1-\gamma)f_s(\theta^2)] \leqslant V_s[\gamma\theta^1 + (1-\gamma)\theta^2] - f_s(\gamma\theta^1 + (1-\gamma)\theta^2)$$

as f_s is a convex function. Therefore $\gamma y^1 + (1-\gamma)y^2 \in M$ and M is a convex set. □

Define $H \equiv \{V\theta - f(\theta) \in \mathcal{R}^S | \theta \in \mathcal{R}^J\}$ and $H_+ \equiv H \cap \mathcal{R}_+^S$, then we have the following two propositions for WAF and SAF concepts, respectively.

PROPOSITION 1. *A security price q is WAF for return V and tax system $\{t_s\}$ if and only if $M \cap \mathcal{R}_+^{1+S} = \{0\} \times H_+$.*

Proof. (Necessary condition) From $y \in M \cap \mathcal{R}_+^{1+S}$ there exists a $\theta \in \mathcal{R}^J$ such that $-q\theta \geqslant 0$ and $0 \leqslant y_s \leqslant V_s\theta - f_s(\theta)$. Because the security price q is WAF for return V and tax system $\{t_s\}$, from $0 \leqslant y_s \leqslant V_s\theta - f_s(\theta)$ we have $q\theta \geqslant 0$. Therefore, $q\theta = 0$.

(Sufficient condition) Suppose there exists a $\theta \in \mathcal{R}^J$ such that $V\theta - f(\theta) \geqslant 0$ and $q\theta < 0$, then $-q\theta > 0$. So $(-q\theta, V\theta - f(\theta)) \in M \cap \mathcal{R}_+^{1+S}$, but $(-q\theta, V\theta - f(\theta)) \notin \{0\} \times H_+$. This is a contradiction! Therefore any portfolio $\theta \in \mathcal{R}^J$ of securities has a positive total cost or gain $q\theta \geqslant 0$ whenever it has a positive payoff in every state $V\theta - f(\theta) \in \mathcal{R}_+^S$. □

PROPOSITION 2. *A security price q is SAF for return V and tax system $\{t_s\}$ if and only if $M \cap \mathcal{R}_+^{1+S} = \{0\}$.*

Proof. (Necessary condition) From $y \in M \cap \mathcal{R}_+^{1+S}$ there exists a $\theta \in \mathcal{R}^J$ such that $-q\theta \geqslant 0$ and $0 \leqslant y_s \leqslant V_s\theta - f_s(\theta)$. If $V\theta - f(\theta) \in \mathcal{R}_+^S \setminus \{0\}$, then from Definition 3(1), $q\theta > 0$. This is a contradiction! If $V_s\theta - f_s(\theta) = 0$ then from Definition 3(2) we have $q\theta \geqslant 0$ hence $-q\theta \leqslant 0$. Thus $q\theta = 0$ and $M \cap \mathcal{R}_+^{1+S} \subseteq \{0\}$, and hence $M \cap \mathcal{R}_+^{1+S} = \{0\}$.

(Sufficient condition) We consider two cases:

(1) If there exists a $\theta \in \mathcal{R}^J$ such that $V\theta - f(\theta) \in \mathcal{R}_+^S \setminus \{0\}$ and $q\theta \leqslant 0$, then $-q\theta \geqslant 0$. Take $y \in M \cap \mathcal{R}_+^{1+S}$ such that $0 \leqslant y_0 = -q\theta$ and $y_s = V_s\theta - f_s(\theta)$ for all s. Then $0 \neq y \in M \cap \mathcal{R}_+^{1+S}$, which is a contradiction. It follows that any portfolio $\theta \in \mathcal{R}^J$ of securities has a positive non-zero total cost or gain $q\theta > 0$ whenever it has a positive non-zero payoff in every state $V\theta - f(\theta) \in \mathcal{R}_+^S \setminus \{0\}$.

(2) If there exists a $\theta \in \mathcal{R}^J$ such that $V\theta - f(\theta) \in \mathcal{R}^S_+$ and $q\theta < 0$, then $-q\theta > 0$. Take $y \in M \cap \mathcal{R}^{1+S}_+$ such that $0 < y_0 = -q\theta$ and $0 \leqslant y_s \leqslant V_s\theta - f_s(\theta)$ for all s. Then $0 \neq y \in M \cap \mathcal{R}^{1+S}_+$, which is a contradiction. Therefore any portfolio $\theta \in \mathcal{R}^J$ of securities has a positive total cost or gain $q\theta \geqslant 0$ whenever it has a positive payoff in every state $V\theta - f(\theta) \in \mathcal{R}^S_+$.

The sufficient condition follows from Definition 3 and the two contradictions. □

Propositions 1 and 2 say that the changes of your wealth today and payoffs tomorrow of any trading strategies cannot both be positive. You cannot get something from nothing. These are the equivalent conditions for WAF and SAF, respectively. These equivalent conditions satisfy convex set separating theorems, which are used to prove Theorems 1 and 2.

THEOREM 1. *A security price q is WAF for return V and tax system $\{t_s\}$ if and only if there exist a strictly positive number $\alpha \in \mathcal{R}_{++}$ and a positive vector $\beta \in \mathcal{R}^S_+$ such that*

$$\sum_{s=1}^{S} \beta_s[V_s\theta - f_s(\theta)] \leqslant \alpha q\theta \quad \textit{for} \quad \theta \in \mathcal{R}^J.$$

Proof. For convenience, we denote $\begin{pmatrix} y_1 \\ \vdots \\ y_S \end{pmatrix}$ as $\hat{y}$.

Both M and $\mathcal{R}^{1+S}_+$ are closed and convex sets. From Proposition 1, a security price q is WAF for return V and tax system $\{t_s\}$ if and only if $M \cap \mathcal{R}^{1+S}_+ = \{0\} \times H_+$. Define $N \equiv \mathcal{R}^{1+S}_+ \setminus (M \cap \mathcal{R}^{1+S}_+) = \mathcal{R}^{1+S}_+ \setminus M$. Note that N is a convex set. Both M and N are non-empty disjoint convex sets $M \cap N = M \cap \mathcal{R}^{1+S}_+ \setminus M = \emptyset$. Obviously, $(1,0) \in \mathcal{R}^{1+S}_+$ and $(1,0) \notin \overline{M-N}$, so $\overline{M-N} \neq \mathcal{R}^{1+S}$. Clark's convex-set separating theorem (Clark, 1994)[4] states that there exists a non-zero continuous linear function $f : \mathcal{R}^{1+S} \to \mathcal{R}$ separating N from M, that is, $f(n) \geqslant 0$ for all $n \in N$ and $f(m) \leqslant 0$ for all $m \in M$. Moreover, $f(1,0) > 0$.

$f : \mathcal{R}^{1+S} \to \mathcal{R}$ is a positive continuous linear function. In fact, for any $e \in \mathcal{R}^{1+S}_+$ and natural number $k = 1, 2, \cdots, e + \left(\frac{1}{k}, 0\right) \in N$ and $\lim_{k\to\infty} e + \left(\frac{1}{k}, 0\right) = e$. Then $f\left(e + \left(\frac{1}{k}, 0\right)\right) \geqslant 0$. Thus $f(e) = \lim_{k\to\infty} f\left(e + \left(\frac{1}{k}, 0\right)\right) \geqslant 0$. That

[4] Clark (1994) presented convex-set separating theorem as follows. Suppose M and N are nonempty disjoint convex sets in a locally convex topological vector space E, then there exists a non-zero continuous linear functional $f : E \to \mathcal{R}$ separating N from M: $f(n) \geqslant 0$ for all $n \in N$ and $f(m) \leqslant 0$ for all $m \in M$ if and only if $\overline{M-N} \neq E$. Moreover, if $\overline{M-N} \neq E$ then for all $e \notin \overline{M-N}$, we may select f so that $f(e) > 0$.

is f is a positive continuous linear function on $\mathcal{R}^{1+S}$. In particular, f is represented by some $\alpha \in \mathcal{R}$ and $\beta \in \mathcal{R}^S$ by $f(y) = \alpha y_0 + \beta\hat{y}$ for any $y \in \mathcal{R}^{1+S}$.

We further prove that $\alpha > 0$ and $\beta \in \mathcal{R}^S_+$. $(1,0) \notin \overline{M-N}$ implies $\alpha = f(1,0) > 0$. If $\beta \notin \mathcal{R}^S_+$, then there exists $\hat{y}_* \in \mathcal{R}^S_+$ such that $\beta\hat{y}_* < 0$. Take $y_{0*} = \frac{1}{2\alpha}(-\beta\hat{y}_*) > 0$, it follows that $y_* \in N$ and $f(y_*) = \alpha y_{0*} + \beta\hat{y}_* = \frac{1}{2}\beta\hat{y}_* < 0$. This is a contradiction. Therefore $\beta \in \mathcal{R}^S_+$.

From $f(m) \leqslant 0$ for all $m \in M$, we have $f(y) = \alpha y_0 + \beta\hat{y} \leqslant 0$ for all $y_0 = -q\theta, \hat{y} \leqslant V\theta - f(\theta)$. Then $\sum_{s=1}^S \beta_s[V_s\theta - f_s(\theta)] \leqslant \alpha q\theta$ for $\theta \in \mathcal{R}^J$. □

If there are no taxes, the result in Theorem 1 reduces to the weak form of the first fundamental theorem in financial economics – the equivalent condition for WAF is that there exists a strictly positive number $\alpha \in \mathcal{R}_{++}$ and a positive vector $\beta \in \mathcal{R}^S_+$ such that $\sum_{s=1}^S \beta_s V_s\theta = \alpha q\theta$ for $\theta \in \mathcal{R}^J$, that is, $q^\top = \frac{1}{\alpha}\beta^\top V$, where $\frac{\beta_s}{\alpha}$ is called "state price" in frictionless markets. The ratio $\frac{\beta_s}{\alpha}$ can be thought as the marginal cost of obtaining an additional unit of account in state s tomorrow. In frictional markets with progress taxtation, we cannot find "state price". However, we can find the minimal cost of obtaining an additional unit of account in state s.

COROLLARY 1. *Suppose the $S \times J$ matrix V is of full rank and $S = J$, denote $V^{-1} = (v_1, \cdots, v_S)$. A security price q is WAF for return V and tax system $\{t_s\}$, then, for $s \in \{1, \cdots, S\}$,*

(1) *there exists a $\delta \leqslant C^1_s$ such that $\frac{\beta_s}{\alpha} \leqslant qv_s$.*
(2) *there exists a $\delta \in [C^k_s, C^{k+1}_s)$, $k = 1, \cdots, K$ such that*

$$\frac{\beta_s}{\alpha\delta}\left[\delta - \sum_{k'=0}^{k-1}(C^{k'+1}_s - C^{k'}_s)t^{k'}_s - (\delta - C^k_s)t^k_s\right] \leqslant qv_s$$

Proof. From Theorem 1, a security price q is WAF for return V and tax system $\{t_s\}$ if and only if there exist a strictly positive number $\alpha \in \mathcal{R}_{++}$ and a positive vector $\beta \in \mathcal{R}^S_+$ such that $\sum_{s=1}^S \beta_s[V_s\theta - f_s(\theta)] \leqslant \alpha q\theta$ for $\theta \in \mathcal{R}^J$. Deleting the sth row V_s of the $S \times J$ matrix V, we have an $(S-1) \times J$ matrix V_{-s}, that is,

$$V_{-s} = \begin{pmatrix} V_1 \\ \vdots \\ V_{s-1} \\ V_{s+1} \\ \vdots \\ V_S \end{pmatrix}$$

for all $s = 1, \cdots, S$. Then the rank of V_{-s} is $J - 1$. Define $\Theta_s \equiv \{\theta \in \mathcal{R}^J \mid V_s\theta = 0\}, s = 1, \cdots, S$. For a given s, there exists a non-zero $\theta_s \in \prod_{s' \neq s} \Theta_{s'}$ such that $V_{-s}\theta_s = 0$. Then

$$\beta_s[V_s\theta_s - f_s(\theta_s)] \leqslant \alpha q \theta_s$$

Since $V^{-1} = (v_1, \cdots, v_S)$, then

$$V_s v_{s'} = \begin{cases} 1, & \text{if } s' = s \\ 0, & \text{if } s' \neq s \end{cases}$$

Take $\theta_s = \delta v_s$, then

$$V_s \theta_{s'} = \begin{cases} \delta, & \text{if } s' = s \\ 0, & \text{if } s' \neq s \end{cases}$$

and $\beta_s[\delta - f_s(\delta v_s)] \leqslant \alpha \delta q v_s$. In particular,

(1) when $\delta < C_s^1$, $f_s(\delta v_s) = 0$, so $\frac{\beta_s}{\alpha} \leqslant q v_s$.
(2) when $C_s^k \leqslant \delta < C_s^{k+1}$ for $k = 1, \cdots, K$

$$f_s(\delta v_s) = \sum_{k'=0}^{k-1} (C_s^{k'+1} - C_s^{k'}) t_s^{k'} + (\delta - C_s^k) t_s^k$$

Therefore,

$$\frac{\beta_s}{\alpha\delta}\left[\delta - \sum_{k'=0}^{k-1} (C_s^{k'+1} - C_s^{k'}) t_s^{k'} - (\delta - C_s^k) t_s^k\right] \leqslant q v_s$$

for $k = 1, \cdots, K$. □

In Theorem 1 and Corollary 1, we only require $\beta^\top \in \mathcal{R}_+^S$. The minimal marginal cost, β_s, may be zero in some state. So there may still exist arbitrage opportunities. We prove next that if a security price is strongly arbitrage-free, the minimal marginal cost of obtaining an additional unit of account in any state is strictly positive.

THEOREM 2. *A security price q is SAF for return V and tax system $\{t_s\}$ if and only if there exist a strictly positive number $\alpha \in \mathcal{R}_{++}$ and a strictly positive vector $\beta \in \mathcal{R}_{++}^S$ such that*

$$\sum_{s=1}^{S} \beta_s[V_s\theta - f_s(\theta)] \leqslant \alpha q \theta \text{ for } \theta \in \mathcal{R}^J.$$

Proof. Both M and $\mathcal{R}_+^{1+S}$ are closed and convex sets. From Proposition 2, a security price q is SAF for return V and tax system $\{t_s\}$ if and only if $M \cap \mathcal{R}_+^{1+S} = \{0\}$. From convex sets separation theorem, there exists a non-zero continuous linear function $f : \mathcal{R}^{1+S} \to \mathcal{R}$ strictly separating $\mathcal{R}_+^{1+S} \setminus \{0\}$ from M, that is, $f(n) > 0$ for all $n \in \mathcal{R}_+^{1+S} \setminus \{0\}$ and $f(m) \leqslant 0$ for all $m \in M$.

The condition $f(n) > 0$ for all $n \in \mathcal{R}_+^{1+S} \setminus \{0\}$ implies that $f : \mathcal{R}^{1+S} \to \mathcal{R}$ is a strictly positive continuous linear function on $\mathcal{R}^{1+S}$. Thus f is represented by some $\alpha > 0$ in $\mathcal{R}$ and $\beta \in \mathcal{R}_{++}^S$ by $f(y) = \alpha y_0 + \beta \hat{y}$ for any $y \in \mathcal{R}_+^{1+S}$.

From $f(m) \leqslant 0$ for all $m \in M$, we have $f(y) = \alpha y_0 + \beta \hat{y} \leqslant 0$ for all $y_0 = -q\theta, \hat{y} \leqslant V\theta - f(\theta)$. Then $\sum_{s=1}^S \beta_s [V_s\theta - f_s(\theta)] \leqslant \alpha q \theta$ for $\theta \in \mathcal{R}^J$. □

If there are no taxes, the result in Theorem 2 reduces to the strong form of the first fundamental theorem in financial economics – the equivalent condition for WAF is that there exists a strictly positive number $\alpha \in \mathcal{R}_{++}$ and a strictly positive vector $\beta \in \mathcal{R}_{++}^S$ such that $\sum_{s=1}^S \beta_s V_s \theta = \alpha q \theta$ for $\theta \in \mathcal{R}^J$, that is $q^\top = \frac{1}{\alpha} \beta^\top V$. Again, $\frac{\beta_s}{\alpha}$ is the strictly positive minimal marginal cost of obtaining an additional unit of account in state s.

COROLLARY 2. *Suppose the $S \times J$ matrix V is of full rank and $S = J$, denote $V^{-1} = (v_1, \cdots, v_S)$. A security price q is SAF for return V and tax system $\{t_s\}$, then, for $s \in \{1, \cdots, S\}$,*

(1) *there exists a $\delta \leqslant C_s^1$ such that $\frac{\beta_s}{\alpha} \leqslant q v_s$.*
(2) *there exists a $\delta \in [C_s^k, C_s^{k+1})$, $k = 1, \cdots, K$ such that*

$$\frac{\beta_s}{\alpha\delta}\left[\delta - \sum_{k'=0}^{k-1}(C_s^{k'+1} - C_s^{k'})t_s^{k'} - (\delta - C_s^k)t_s^k\right] \leqslant q v_s$$

Proof. The proof is the same as Corollary 1. □

Remark. In frictionless case, the weakly and strongly arbitrage-free asset pricing is equivalent to that there is a state price $\beta^* \in \mathcal{R}_+^S$ for weak concept and $\beta^* \in \mathcal{R}_{++}^S$ for strict concept such that $q = \beta^* V$, that is, $q v_s = \beta_s^*$. The result is a special example of our Corollaries 1 and 2.

4. Existence of SF Equilibrium

PROPOSITION 3. *If the preference relation $\succeq_i$ is strictly monotone for $i = 1, \cdots, I$ and (p, q) is a FM equilibrium price system, then the security price q is SAF for return V and tax system $\{t_s\}$.*

Now we begin to prove the existence of spot-financial market equilibrium. Lemma 2 studies the budget correspondence. Lemma 3 studies demand correspondence. Lemma 4 is Debreu–Gale–Nikido Lemma in this case. Theorem 3 is one of the main results in this paper.

The budget set of agent i is given by the budget correspondence $B_i : \mathcal{R}_+^{L(1+S)} \times \mathcal{R}_+^J \to \mathcal{R}_+^{L(1+S)} \times \mathcal{R}^J$ as follows:

$$B_i(p,q) = \left\{ (x,\theta) \in \mathcal{R}_+^{L(1+S)} \times \mathcal{R}^J \middle| \begin{array}{l} p_0(x_0 - e_0^i) + q\theta \leqslant 0 \\ p_s(x_s - e_s^i) \leqslant V_s\theta - f_s(\theta) + \lambda_i G_s, \quad s = 1, \cdots, S \end{array} \right\}$$

LEMMA 2. *If $e^i \in \mathcal{R}_{++}^{L(1+S)}$, then the budget correspondence B_i satisfies*

(1) *B_i is a closed correspondence;*
(2) *$B_i(p,q)$ is a closed and convex set in $\mathcal{R}_+^{L(1+S)} \times \mathcal{R}_+^J$;*
(3) *$B_i(p,q)$ is a compact set for $p \in \mathcal{R}_{++}^{L(1+S)}$ and $q \in \mathcal{R}_{++}^J$;*
(4) *B_i is lower hemi-continuous for $p \in \mathcal{R}_{++}^{L(1+S)}$.*

Proof. (1) Let $(x^n, \theta^n) \in B_i(p^n, q^n)$ with $\lim_{n\to\infty}(x^n, \theta^n) = (x, \theta)$ and $\lim_{n\to\infty}(p^n, q^n) = (p, q)$. Then $(x^n, \theta^n) \in \mathcal{R}_+^{L(1+S)} \times \mathcal{R}^J$ and

$$\begin{cases} p_0^n(x_0^n - e_0^i) + q^n\theta^n \leqslant 0 \\ p_s^n(x_s^n - e_s^i) \leqslant V_s\theta^n - f_s(\theta^n) + \lambda_i G_s, \quad s = 1, \cdots, S \end{cases}$$

Thus $(x, \theta) \in \mathcal{R}_+^{L(1+S)} \times \mathcal{R}^J$ and

$$\begin{cases} p_0(x_0 - e_0^i) + q\theta \leqslant 0 \\ p_s(x_s - e_s^i) \leqslant V_s\theta - f_s(\theta) + \lambda_i G_s, \quad s = 1, \cdots, S \end{cases}$$

That is, $(x, \theta) \in B_i(p, q)$.

(2) It is obvious that $B_i(p,q)$ is a closed set in $\mathcal{R}_+^{L(1+S)} \times \mathcal{R}_+^J$. Now we prove that $B_i(p,q)$ is a convex set in $\mathcal{R}_+^{L(1+S)} \times \mathcal{R}_+^J$. Suppose $(x^n, \theta^n) \in B_i(p,q)$ for $n = 1, 2$, then $(x^n, \theta^n) \in \mathcal{R}_+^{L(1+S)} \times \mathcal{R}^J$ and

$$\begin{cases} p_0(x_0^n - e_0^i) + q\theta^n \leqslant 0 \\ p_s(x_s^n - e_s^i) \leqslant V_s\theta^n - f_s(\theta^n) + \lambda_i G_s, \quad s = 1, \cdots, S \end{cases}$$

For $\gamma \in [0, 1]$, we have

$$\begin{cases} p_0\{[\gamma x_0^1 + (1-\gamma)x_0^2] - e_0^i\} + q[\gamma\theta^1 + (1-\gamma)\theta^2] \leqslant 0 \\ p_s\{[\gamma x_s^1 + (1-\gamma)x_s^2] - e_s^i\} \leqslant V_s[\gamma\theta^1 + (1-\gamma)\theta^2] - [\gamma f_s(\theta^1) + (1-\gamma)f_s(\theta^2)] + \lambda_i G_s \end{cases}$$

$f_s(\theta)$ is a convex function, so we have $\gamma f_s(\theta^1) + (1-\gamma)\ f_s(\theta^2) \geqslant f_s(\gamma\theta^1 + (1-\gamma)\theta^2)$. Thus $p_s\{[\gamma x_s^1 + (1-\gamma)x_s^2] - e_s^i\} \leqslant V_s[\gamma\theta^1 + (1-\gamma)\theta^2] - f_s(\gamma\theta^1 + (1-\gamma)\theta^2) + \lambda_i G_s$ for $s = 1, \cdots, S$. Therefore, $(\gamma x^1 + (1-\gamma)x^2, \gamma\theta^1 + (1-\gamma)\theta^2) \in B_i(p,q)$. So $B_i(p,q)$ is a convex set in $\mathcal{R}_+^{L(1+S)} \times \mathcal{R}_+^J$.

(3) Let $(x^n, \theta^n) \in B_i(p,q)$ for $(p,q) \in \mathcal{R}_+^{L(1+S)} \times \mathcal{R}_+^J$, then $(x^n, \theta^n) \in \mathcal{R}_+^{L(1+S)} \times \mathcal{R}^J$ and

$$p_0(x_0^n - e_0^i) + q\theta^n \leqslant 0 \tag{1}$$

$$p_s(x_s^n - e_s^i) \leqslant V_s\theta^n - f_s(\theta^n) + \lambda_i G_s, \quad s = 1, \cdots, S \tag{2}$$

From Theorem 2, a security price q is SAF for return V and tax system $\{t_s\}$ if and only if there exist a strictly positive number $\alpha \in \mathcal{R}_{++}$ and a strictly positive vector $\beta \in \mathcal{R}^S_{++}$ such that $\sum_{s=1}^S \beta_s[V_s\theta - f_s(\theta)] \leqslant \alpha q\theta$ for $\theta \in \mathcal{R}^J$. From $\alpha \times (1) + \sum_{s=1}^S \beta_s \times (2)$, we have

$$\alpha p_0(x_0^n - e_0^i) + \sum_{s=1}^S \beta_s p_s(x_s^n - e_s^i) \leqslant -\alpha q\theta^n + \sum_{s=1}^S \beta_s[V_s\theta^n - f_s(\theta^n) + \lambda_i G_s] \leqslant \lambda_i \sum_{s=1}^S G_s$$

So

$$\alpha p_0 x_0^n + \sum_{s=1}^S \beta_s p_s x_s^n \leqslant \alpha p_0 e_0^i + \sum_{s=1}^S \beta_s p_s e_s^i + \lambda_i \sum_{s=1}^S G_s$$

Thus $\{x^n\}$ is bounded. From (1), $q\theta^n \leqslant p_0 e_0^i$, so $\{\theta^n\}$ is bounded from above. From (2) $-p_s e_s^i \leqslant V_s\theta^n - f_s(\theta^n) + \lambda_i G_s$, so $\{\theta^n\}$ is bounded from below. Thus $\{\theta^n\}$ is bounded. Since $\{(x^n, \theta^n)\}$ is bounded in $\mathcal{R}^L_+ \times \mathcal{R}^J$, without loss of generality, then $\{(x^n, \theta^n)\}$ converges to (x, θ), which belongs to $B_i(p, q)$. Therefore, $B_i(p, q)$ is a compact set in $\mathcal{R}^{L(1+S)}_{++} \times \mathcal{R}^J_{++}$.

(4) The method of proof is from Hildenbrand (1974). Consider the correspondence $B_i^0 : \mathcal{R}^{L(1+S)}_+ \times \mathcal{R}^J_+ \to \mathcal{R}^{L(1+S)}_+ \times \mathcal{R}^J$ defined by

$$B_i^0(p, q) = \left\{ (x, \theta) \in \mathcal{R}^{L(1+S)}_+ \times \mathcal{R}^J \,\middle|\, \begin{array}{l} p_0(x_0 - e_0^i) + q\theta < 0 \\ p_s(x_s - e_s^i) < V_s\theta - f_s(\theta) + \lambda_i G_s, \quad s = 1, \cdots, S \end{array} \right\}$$

For $p \in \mathcal{R}^{L(1+S)}_{++}$, $(0, 0) \in B_i^0(p, q)$ from $e^i \in \mathcal{R}^{L(1+S)}_{++}$, then $B_i^0(p, q)$ is nonempty. Let $(p^n, q^n) \in \mathcal{R}^{L(1+S)}_{++} \times \mathcal{R}^J$ with $\lim_{n\to\infty}(p^n, q^n) = (p, q)$ where $(x, \theta) \in B_i^0(p, q)$ for $(x, \theta) \in \mathcal{R}^{L(1+S)}_+ \times \mathcal{R}^J_+$, then

$$\begin{cases} p_0(x_0 - e_0^i) + q\theta < 0 \\ p_s(x_s - e_s^i) < V_s\theta - f_s(\theta) + \lambda_i G_s, \quad s = 1, \cdots, S \end{cases}$$

Then for every $\{(x^n, \theta^n)\}$ with $\lim_{n\to\infty}(x^n, \theta^n) = (x, \theta)$ and for n large enough,

$$\begin{cases} p_0^n(x_0^n - e_0^i) + q^n\theta^n < 0 \\ p_s^n(x_s^n - e_s^i) < V_s\theta^n - f_s(\theta^n) + \lambda_i G_s, \quad s = 1, \cdots, S \end{cases}$$

Thus $(x^n, \theta^n) \in B_i^0(p^n, q^n)$ for n large enough, which implies that B_i^0 is lower hemi-continuous for $p \in \mathcal{R}^{L(1+S)}_{++}$. Since the closure of lower hemi-continuous correspondence is also lower hemi-continuous, (4) follows. □

Define the individual demand correspondence of agent i by $\phi_i : \mathcal{R}^{L(1+S)}_+ \times \mathcal{R}^J_+ \to \mathcal{R}^{L(1+S)}_+ \times \mathcal{R}^J$:

$$\phi_i(p, q) = \{(x, \theta) \in B_i(p, q) |\text{ there is no } (x', \theta') \in B_i(p, q) \text{ with } x' \succ_i x\}$$

The demand correspondence ϕ_i has the following properties:

LEMMA 3. *Under the Preference Assumption, if* $e^i \in \mathcal{R}^{L(1+S)}_{++}$, *then*

(1) ϕ_i *is nonempty, compact and convex valued*;
(2) ϕ_i *is upper hemi-continuous for* $p \in \mathcal{R}^{L(1+S)}_{++}$ *and* $q \in \mathcal{R}^J_{++}$;
(3) ϕ_i *is a closed correspondence for* $p \in \mathcal{R}^{L(1+S)}_{++}$ *and* $q \in \mathcal{R}^J_{++}$;

(4) *if the sequence* $\{p^n\}$ *in* $\mathcal{R}_{++}^{L(1+S)}$ *converges to p which is not strictly positive* (*that is*, $p \in \partial\mathcal{R}_{++}^{L(1+S)} \setminus \{0\}$), *then*

$$\lim_{n\to\infty} \inf\{\|x\| \,|(x,\theta) \in \phi_i(p^n, q) \textit{ for some } \theta\} = +\infty.$$

Proof. Consider the preference relation $\succeq_i'$ defined on $B_i(p,q)$ by $(x_1,\theta_1) \succeq_i' (x_2,\theta_2)$ if $x_1 \succeq_i x_2$, then $\succeq_i'$ is a continuous, weakly convex, strictly monotone, and complete preference relation on a compact subset $B_i(p,q)$ of $\mathcal{R}_+^{L(1+S)} \times \mathcal{R}^J$.

(1) The proof is from Aliprantis et al. (1989).

For each $(x,\theta) \in B_i(p,q)$, let

$$C_{(x,\theta)} = \{(x',\theta') \in B_i(p,q) \mid (x',\theta') \succeq_i (x,\theta)\}$$

Since $\succeq_i'$ is upper hemi-continuous, the non-empty set $C_{(x,\theta)}$ is closed and hence compact. Now note the set of all maximal elements of $\succeq_i'$ is the compact set $\bigcap_{(x,\theta)\in B_i(p,q)} C_{(x,\theta)}$. We shall show that

$$\bigcap_{(x,\theta)\in B_i(p,q)} C_{(x,\theta)} \neq \emptyset$$

To this end, let $(x_1,\theta_1),(x_2,\theta_2),\cdots,(x_N,\theta_N) \in B_i(p,q)$. Since $\succeq_i'$ is a complete binary relation, the set $\{(x_1,\theta_1),(x_2,\theta_2),\cdots,(x_N,\theta_N)\}$ is completely ordered. We can assume that $(x_1,\theta_1) \succeq_i' (x_2,\theta_2) \succeq_i' \cdots \succeq_i' (x_N,\theta_N)$. This implies $C_{(x_1,\theta_1)} \subseteq C_{(x_2,\theta_2)} \subseteq \cdots \subseteq C_{(x_N,\theta_N)}$, and so $\bigcap_{n=1}^N C_{(x_n,\theta_n)} = C_{(x_1,\theta_1)} \neq \emptyset$. Thus the collection of closed sets $\{C_{(x,\theta)}|(x,\theta) \in B_i(p,q)\}$ has the finite intersection property. By the compactness of $B_i(p,q)$, the set $\bigcap_{(x,\theta)\in B_i(p,q)} C_{(x,\theta)}$ is non-empty.

Let (x_1,θ_1) and (x_2,θ_2) be two maximal elements of $\succeq_i'$ in $B_i(p,q)$ and let $0 < \alpha < 1$. Then $\alpha(x_1,\theta_1) + (1-\alpha)(x_2,\theta_2) \in B_i(p,q)$ and by the convexity of $\succeq_i'$, we see that $\alpha(x_1,\theta_1) + (1-\alpha)(x_2,\theta_2) \succeq_i' (x_1,\theta_1)$. On the other hand, by the maximality of (x_1,θ_1), we have that $(x_1,\theta_1) \succeq_i' \alpha(x_1,\theta_1) + (1-\alpha)(x_2,\theta_2)$ and therefore $\alpha(x_1,\theta_1) + (1-\alpha)(x_2,\theta_2)$ is also a maximal element of $\succeq_i'$.

(2) and (3) Let $(x^n,\theta^n) \in \phi_i(p^n,q^n)$ with $\lim_{n\to\infty}(x^n,\theta^n) = (x,\theta)$ and $\lim_{n\to\infty}(p^n,q^n) = (p,q)$, then $(x,\theta) \in B_i(p,q)$. We need to prove that (x,θ) is a maximal element for $\succeq_i'$ in $B_i(p,q)$. Let $(x',\theta') \in B_i(p,q)$, then

$$\begin{cases} p_0(x_0' - e_0^i) + q\theta' \leqslant 0 \\ p_s(x_s' - e_s^i) \leqslant V_s\theta' - f_s(\theta') + \lambda_i G_s, \quad s = 1,\cdots,S \end{cases}$$

For each $0 < \gamma < 1$ we have

$$\begin{cases} p_0(\gamma x_0' - e_0^i) + q\theta' < 0 \\ p_s(\gamma x_s' - e_s^i) < V_s\theta' - f_s(\theta') + \lambda_i G_s, \quad s = 1,\cdots,S \end{cases}$$

From $\lim_{n\to\infty}(p^n,q^n) = (p,q)$, we see that there exists some n_0 such that for all $n > n_0$ we have

$$\begin{cases} p_0^n(\gamma x'_0 - e_0^i) + q^n\theta' < 0 \\ p_s^n(\gamma x'_s - e_s^i) < V_s\theta' - f_s(\theta') + \lambda_i G_s, \quad s = 1, \cdots, S \end{cases}$$

Thus $(x^n, \theta^n) \succeq'_i (\gamma x', \theta')$ for all $n > n_0$. From the continuity of $\succeq'_i$, we have $(x, \theta) \succeq'_i (\gamma x', \theta')$. Let $\gamma \uparrow 1$, we have $(x, \theta) \succeq'_i (x', \theta')$. Therefore, $(x, \theta) \in \phi_i(p, q)$.

(4) Assume there is a subsequence $\{x^n\}$ such that $\lim_{n\to\infty} x^n = x$ and $(x^n, \theta^n) \in \phi_i(p^n, q^n)$ for some θ^n. From the strict monotonicity of $\succeq'_i$ and $(x^n, \theta^n) \in \phi_i(p^n, q^n)$ we have

$$\begin{cases} p_0^n(x_0^n - e_0^i) = -q^n\theta^n \\ p_s^n(x_s^n - e_s^i) = V_s\theta^n - f_s(\theta^n) + \lambda_i G_s, \quad s = 1, \cdots, S \end{cases}$$

Take the limit as $n \to \infty$, we have

$$\begin{cases} p_0(x_0 - e_0^i) = -q\theta \\ p_s(x_s - e_s^i) = V_s\theta - f_s(\theta) + \lambda_i G_s, \quad s = 1, \cdots, S \end{cases}$$

and $\theta^n \to \theta$ for some $\theta \in \mathcal{R}^J$. By (3) ϕ_i is closed for $p \in \mathcal{R}^L_{++}$, therefore $(x, \theta) \in \phi_i(p, q)$. On the other hand, since p is not strictly positive and every commodity is desired, so $\phi_i(p, q) = \emptyset$. Contradiction! □

Define total excess–demand correspondence $\phi : \mathcal{R}_+^{L(1+S)} \times \mathcal{R}_+^J \to \mathcal{R}^{L(1+S)} \times \mathcal{R}^J$ by

$$\phi(p, q) = \sum_{i=1}^{I} \phi_i(p, q) - \sum_{i=1}^{I} (e^i, 0)$$

PROPOSITION 4. *Under the Preference Assumption, if $e^i \in \mathcal{R}_{++}^{L(1+S)}, i \in \mathcal{I}$, then*

(1) *ϕ is non-empty, compact and convex valued;*
(2) *ϕ is upper hemi-continuous for $p \in \mathcal{R}_{++}^{L(1+S)}$ and $q \in \mathcal{R}_{++}^J$;*
(3) *ϕ is a closed correspondence for $p \in \mathcal{R}_{++}^{L(1+S)}$ and $q \in \mathcal{R}_{++}^J$;*
(4) *if the sequence $\{p^n\}$ in $\mathcal{R}_{++}^{L(1+S)}$ converges to p which is not strictly positive (that is, $p \in \partial\mathcal{R}_{++}^{L(1+S)} \setminus \{0\}$), then*

$$\lim_{n\to\infty} \inf\{\|z\| \mid (z, \theta) \in \phi(p^n, q) \text{ for some } \theta\} = +\infty.$$

If $0 \in \phi(p, q)$, then clearly (p, q) is an equilibrium price system. ϕ satisfies the following Walras' Law: for every $p \in \mathcal{R}_{++}^{L(1+S)}$, $q \in \mathcal{R}_{++}^J$ and $(z, \theta) \in \phi(p, q)$, then

$$p_0 z_0 + q\theta = 0$$

$$p_s z_s = V_s\theta, \quad s = 1, \cdots, S$$

Consider the following price sets:

$$\Delta = \left\{ (p_0, q) \in \mathcal{R}_+^L \times \mathcal{R}_+^J \,\middle|\, \sum_{l=1}^{L} p_{0l} + \sum_{j=1}^{J} q_j = 1 \right\}$$

$$P_s = \left\{ p_s \in \mathcal{R}^L_+ \middle| \sum_{l=1}^{L} p_{sl} = 1 \right\}$$

$$T = \Delta \times \prod_{s=1}^{S} P_s$$

$$\Delta^n = \left\{ (p_0, q) \in \Delta \middle| p_{0l} \geqslant \frac{1}{n} \text{ and } q_j \geqslant \frac{1}{n} \text{ for all } l = 1, \cdots, L,\ j = 1, \cdots, J \right\}, \quad n \geqslant L + J$$

$$P^n_s = \left\{ p_s \in P_s \middle| p_{sl} \geqslant \frac{1}{n}, l = 1, \cdots, L \right\}, \quad n \geqslant L$$

$$T^n = \Delta^n \times \prod_{s=1}^{S} P^n_s, \quad n \geqslant L + J$$

Clearly all these sets are compact and convex, and furthermore

$$\operatorname{int} \Delta = \bigcup_n \Delta^n$$

$$\operatorname{int} P_s = \bigcup_n P^n_s$$

$$\operatorname{int} T = \operatorname{int} \Delta \times \prod_{s=1}^{S} \operatorname{int} P_s \qquad .$$

LEMMA 4. *Suppose that the preference assumption holds. Let* $e^i \in \mathcal{R}^{L(1+S)}_{++}$, $i \in \mathcal{I}$. *For every* n *there exist* $(p^n, q^n) \in T^n$ *and* $(z^n, \theta^n) \in \phi(p^n, q^n)$ *such that for all* $(p, q) \in T^n$,

$$p_0 z^n_0 + q\theta^n \leqslant 0$$
$$p_s z^n_s \leqslant V_s \theta^n, \quad s = 1, \cdots, S$$

Proof. Let Z^n be a compact and convex set such that $\phi(T^n) \subseteq Z^n$. For every $(z, \theta) \in Z^n$, we consider a correspondence $\mu^n : Z^n \to T^n$ by

$$\mu^n(z, \theta) = \left\{ (p, q) \in T^n \middle| \begin{array}{l} p_0 z_0 + q\theta = \max\limits_{(p'_0, q')} p'_0 z_0 + q'\theta \\ p_s z_s = \max\limits_{p'_s} p'_s z_s, \quad s = 1, \cdots, S \end{array} \right\}$$

$\mu^n(z,\theta)$ is a non-empty convex subset of T^n for any $(z,\theta) \in Z^n$ and μ^n is a closed correspondence. Thus the correspondence

$$\mu^n \times \phi\colon Z^n \times T^n \to Z^n \times T^n$$

satisfies the conditions of Kakutani's fixed point theorem. Then there exists a fixed point $((z^n,\theta^n),(p^n,q^n))$ of $\mu^n \times \phi$, that is, $((z^n,\theta^n),(p^n,q^n)) \in (\mu^n \times \phi)((z^n,\theta^n),(p^n,q^n))$, both $(z^n,\theta^n) \in \phi(p^n,q^n)$ and $(p^n,q^n) \in \mu^n(z^n,\theta^n)$. From $(p^n,q^n) \in \mu^n(z^n,\theta^n)$, we have, for all $(p,q) \in T^n$,

$$p_0 z_0^n + q\theta^n \leqslant p_0^n z_0^n + q^n\theta^n$$

$$p_s z_s^n \leqslant p_s^n z_s^n, \quad s = 1,\cdots,S$$

From $(z^n,\theta^n) \in \phi(p^n,q^n)$, we have $(z^n + \sum_{i=1}^I e^i, \theta^n) \in \sum_{i=1}^I \phi_i(p^n,q^n)$. Since $\theta^n = \sum_{i=1}^I \theta^{in}$ and $z^n = \sum_{i=1}^I z^{in}$,

$$p_0^n z_0^{in} + q^n\theta^{in} = 0$$

$$p_s^n z_s^{in} = V_s\theta^{in} - f_s(\theta^{in}) + \lambda_i \sum_{i'=1}^{I} f_s(\theta^{i'n}), \quad s = 1,\cdots,S$$

Sum over i we have

$$p_0^n z_0^n + q^n\theta^n = 0$$

$$p_s^n z_s^n = V_s\theta^n, \quad s = 1,\cdots,S$$

Therefore we have

$$p_0 z_0^n + q\theta^n \leqslant 0$$

$$p_s z_s^n \leqslant V_s\theta^n, \quad s = 1,\cdots,S$$

□

THEOREM 3. *Under the preference assumption, if $e^i \in \mathcal{R}_{++}^{L(1+S)}$ for $i \in \mathcal{I}$, security price q is SAF for return V and tax system $\{t_s\}$, then there exists a SF equilibrium $((\bar{x}^i,\bar{\theta}^i),(\bar{p},\bar{q}))$ with $\bar{p} \in \mathcal{R}_{++}^{L(1+S)}$ and $\bar{q} \in \mathcal{R}_{++}^J$.*

Proof. Consider the sequences $\{(p^n,q^n)\}$, $\{(z^n,\theta^n)\}$ from Lemma 4. Since $\{(p^n,q^n)\} \in T^n \subseteq T$, $\{(p^n,q^n)\}$ has a convergent subsequence. Without loss of generality we may assume that $\{(p^n,q^n)\}$ is convergent to $(\bar{p},\bar{q})$.

$$\lim_{n\to\infty} (p^n,q^n) = (\bar{p},\bar{q}) \in T$$

Now we show that $\{z^n\}$ is bounded. From Theorem 2, a security price q is SAF for return V and tax system $\{t_s\}$ if and only if there exist a strictly positive number $\alpha \in \mathcal{R}_{++}$ and a strictly positive vector $\beta \in \mathcal{R}_{++}^S$ such that $\sum_{s=1}^S \beta_s[V_s\theta - f_s(\theta)] \leqslant \alpha q\theta$ for $\theta \in \mathcal{R}^J$.

Let

$$D = \left\{ p \in \mathcal{R}_+^{L(1+S)} \,\middle|\, \sum_{s=0}^{S}\sum_{l=1}^{L} p_{sl} = 1 \right\}$$

For every $p \in \operatorname{int} D$, we can take $q \in \operatorname{int}\{q \in \mathcal{R}_+^J | \sum_{j=1}^J q_j = \sum_{s=1}^S \sum_{l=1}^L p_{sl}\}$ such that $(p_0, q) \in \operatorname{int}\Delta$ and $\frac{p_s}{\sum_{l=1}^L p_{sl}} \in \operatorname{int} P_s, s = 1, \cdots, S$. There exists on integer N such that when $n \geqslant N$, $(p_0, q) \in \Delta^n$ and $\frac{p_s}{\sum_{l=1}^L p_{sl}} \in P_s^n, s = 1, \cdots, S$. From Lemma 4, we have

$$p_0 z_0^n + q\theta^n \leqslant 0 \tag{3}$$

$$\frac{p_s}{\sum_{l=1}^L p_{sl}} z_s^n \leqslant V_s \theta^n, \quad s = 1, \cdots, S \tag{4}$$

From relations (3) and (4), sequence $\{\theta^n\}$ is bounded. $\alpha \times (3) + \sum_{s=1}^S \beta_s \times (4)$ we have

$$\alpha p_0 z_0^n + \sum_{s=1}^S \beta_s \frac{p_s}{\sum_{l=1}^L p_{sl}} z_s^n \leqslant -\alpha q \theta^n + \sum_{s=1}^S \beta_s V_s \theta^n \leqslant \sum_{s=1}^S \beta_s f_s(\theta^n)$$

from SAF proposition.

Thus $\{z^n\}$ is bounded. Therefore, it has a convergent subsequence. Without loss of generality we may assume that $\{z^n\}$ converges to some $\bar{z}$, $\lim_{n\to\infty} z^n = \bar{z}$.

Note that $\lim_{n\to\infty}(p^n, q^n) = (\bar{p}, \bar{q})$. Since $(z^n, \theta^n) \in \phi(p^n, q^n)$, we obtain from Walras' Law that

$$p_0^n z_0^n + q^n \theta^n = 0 \tag{5}$$

$$p_s^n z_s^n = V_s \theta^n, \quad s = 1, \cdots, S \tag{6}$$

Taking limit as $n \to \infty$, we obtain θ^n converges some $\bar{\theta}$, $\lim_{n\to\infty} \theta^n = \bar{\theta}$.

We claim now that $\bar{p} \in \mathcal{R}_{++}^{L(1+S)}$ from Proposition 4(4).

By closeness of ϕ we have $(\bar{z}, \bar{\theta}) \in \phi(p, q)$. To complete the proof we have to show that $(\bar{z}, \bar{\theta}) = 0$. Taking limit of (3) as $n \to \infty$, we obtain $p_0\bar{z}_0 + q\bar{\theta} \leqslant 0$, so $\bar{z_0} \leqslant 0$, $\bar{\theta} \leqslant 0$. Taking limit of (5) we get $\bar{p}_0\bar{z}_0 + \bar{q}\bar{\theta} = 0$. So $\bar{z}_0 = 0$, $\bar{\theta} = 0$. Taking limit of (4) and (6) we have $p_s\bar{z}_s \leqslant V_s\bar{\theta} = 0$ and $\bar{p}_s\bar{z}_s = V_s\bar{\theta} = 0$. So $\bar{z}_s = 0$. □

Remark. If the government does not transfer tax revenues to consumers, we can prove the existence of general equilibrium defined in footnote 3.

5. Concluding Remarks

This paper studies no-arbitrage asset pricing and the existence of general equilibrium under progressive anonymous taxation in a two-period model. We provide no-arbitrage conditions on tax rules and asset returns. Prices exclude arbitrages along all the investors' marginal tax rates. Under usual assumptions of preferences and endowments, we prove the existence of general equilibrium. Werner (1985) proves the existence of general equilibrium for two-period economies. Duffie (1987) extends the existence results to multi-period cases. Florenzano and Gourdel (1994) check the existence of stochastic equilibrium. Our paper extends their results to incomplete frictional financial markets. As in the frictionless case, our results (Theorems 1, 2 and 3) hold in multi-period settings.

In this paper, we give the form of the tax function f_s, $s = 1, \cdots, S$. We only use the properties of continuity and convexity of the tax function when we prove Theorems 1, 2 and 3. The specific form of the tax function is not needed in the process of proofs. Only Corollaries 1 and 2 depend on the form of the tax function.

6. Acknowledgements

I would like to thank Srihari Govindan, Arthur Robson, Peter Streufert, Shunming Zhang, and two anonymous referees for their helpful comments. I am also grateful to seminar participants at the University of Western Ontario and 37th Annual Meeting of the Canadian Economics Association. All errors are mine.

References

1. Aliprantis, C.D., Brown, D.J. and Burkinshaw, O. (1989), *Existence and Optimality of Competitive Equilibria*, Springer-Verlag, Berlin, Heidelberg.
2. Basak, S. and Croitoru, B. (2001), Non-linear taxation, tax-arbitrage and equilibrium asset prices. *Journal of Mathematical Economics* 35, 347–382.
3. Chen, Z. (1995), Financial innovation and arbitrage pricing in frictional economics. *Journal of Economic Theory* 65, 117–135.
4. Clark, S.A. (1993), The valuation problem in arbitrage pricing theory. *Journal of Mathematical Economics* 22, 463–478.
5. Clark, S.A. (1994), Vector space methods in additive theory. *Journal of Mathematical Economics*, conditionally accepted.
6. Dammon, R.M. and Green, R.C. (1987), Tax arbitrage and the existence of equilibrium prices for financial assets. *Journal of Finance* 42, 1143–1166.
7. Duffie, D. (1987), Stochastic equilibria with incomplete financial markets. *Journal of Economic Theory* 41, 405–416.
8. Duffie, D. (2001), *Dynamic Asset Pricing Theory*, 3rd ed., Princeton University Press, Princeton, NJ.
9. Florenzano, M. and Gourdel, P. (1994), T-period economies with incomplete markets. *Economics Letters* 44, 91–97.
10. Harrison, J.M. and Kreps, D.M. (1979), Martingales and arbitrage in multi-period security markets. *Journal of Economic Theory* 20, 381–408.
11. Hildenbrand, W. (1974), *Core and Equilibria of A Large Economy*, Princeton University Press, Princeton, NJ.

12. Jones, C. and Milne, F. (1992), Tax arbitrage, existence of equilibrium, and bounded tax rebates. *Mathematical Finance* 2, 189–196.
13. Jouini, E. (2001), Arbitrage and control problems in finance. A presentation. *Journal of Mathematical Economics* 35, 167–183.
14. Jouini, E. and Kallal, H. (1995a), Martingales and arbitrage in securities markets with transaction costs. *Journal of Economic Theory* 66, 178–197.
15. Jouini, E. and Kallal, H. (1995b), Arbitrage in securities markets with short-sales constraints. *Mathematical Finance* 5, 197–232.
16. Jouini, E., Kallal, H. and Napp, C. (2001), Arbitrage and viability in securities markets with fixed trading costs. *Journal of Mathematical Economics* 35, 197–221.
17. Kabanov, Y.M. and Stricker, C. (2001), The Harrison–Pliska arbitrage pricing theorem under transaction costs. *Journal of Mathematical Economics* 35, 185–196.
18. Kreps, D.M. (1981), Arbitrage and equilibrium in economies with infinitely many commodities. *Journal of Mathematical Economics* 8, 15–35.
19. Magill, M. and Shafer, W. (1991), Incomplete markets. In: Hildenbrand, W. and Sonnenschein, H. (eds.), *Handbook of Mathematical Economics*, Vol. IV, pp. 1523–1614. Elsevier Science Publisher, Amsterdam.
20. Pham, H. and Touzi, N. (1999), The fundamental theorem of asset pricing with cone constraints. *Journal of Mathematical Economics* 31, 265–279.
21. Werner, J. (1985), Equilibrium in economies with incomplete financial markets. *Journal of Economic Theory* 36, 110–119.
22. Zhang, S., Xu, C. and Deng, X. (2002), Dynamic arbitrage-free asset pricing with proportional transaction costs. *Mathematical Finance* 12, 89–97.